The Infamous GCSE Question

In the summer of 2022, Edexcel released an abomination of a question on their HT Math Paper 1, sending students home crying and complaining to their teachers. It is often referred to as the hardest GCSE Maths question of all time.

In this article, I will go over all the different methods that can be used to answer this question. In hindsight, this is just an exercise to improve my LaTex skills. However, I hope this is informative.

Now, feast your eyes upon the beast below.

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The diagram shows three circles, each of radius 4cm.

The centres of the circles are A,B,and C such that ABC is a straight line and $AB = BC = 4$cm.
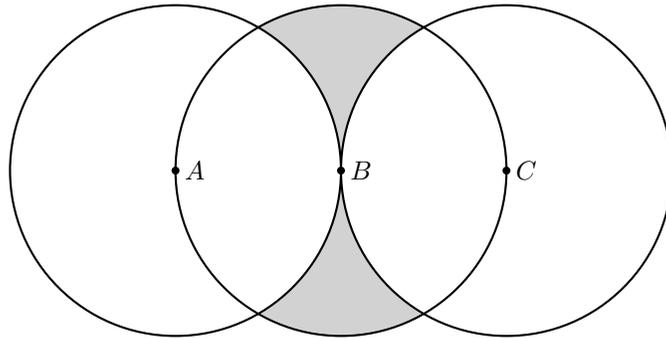
Work out the total area of the two shaded regions.
Give your answer in terms of π.

(5 marks)

Methods

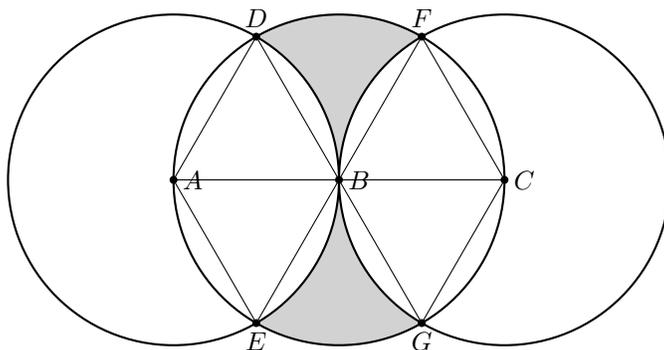Looks nasty right? Below are the different methods I will cover.

1. Geometric (The GCSE method)
2. Polar Integration (r, θ)
3. Integration (Trig Sub)

1. Geometric

The rule of thumb for a geometry question made by Edexcel is always look for **triangles**.

Let us begin by drawing lines to connect intersections and centres to form triangles.

Let the intersection points be D, E, F, and G.

All lines drawn were from a circle's centre to circumference. Therefore, they are all radii (length 4cm).

Which means that 4 equilateral triangles of side length 4cm have been formed. Which also means that all angles in these triangles are $60°$.

To find the shaded area, we will only focus on the upper shaded area (within sector DBF), and double our result at the end.

Notice that after drawing these lines, we have formed not only triangles, but sectors and segments.

From this we can see that:

$$\frac{1}{2}A = (\text{Area Sector}) - 2(\text{Area Segment})$$

To proceed with this calculation, we will need the following prior knowledge:

$$A_{\text{Sector}} = \pi r^2 \times \frac{\theta}{360}$$

$$A_{\text{Equilateral} \triangle} = \frac{\sqrt{3}}{4}s^2$$

$$A_{\text{Segment}} = A_{\text{Sector}} - A_{\triangle}$$

Now just finishing up.

$$\frac{1}{2}A = \pi(4)^2 \times \frac{60}{360} - 2\left(\pi(4)^2 \times \frac{60}{360} - \frac{\sqrt{3}}{4}(4)^2\right)$$

$$= \frac{16\pi}{6} - 2\left(\frac{16\pi}{6} - 4\sqrt{3}\right)$$

$$= \frac{8\pi}{3} - \frac{16\pi}{3} + 8\sqrt{3}$$

$$= 8\sqrt{3} - \frac{8\pi}{3}$$

$$A = \boxed{16\sqrt{3} - \frac{16\pi}{3} \text{ cm}^2}$$

This is the easiest and most efficient method, but by far the most boring.

Now, onto the fun stuff.

2. Polar Integration

It is absurd how a Year 13 method is applicable to a Year 11 question, but it truly underscores the absolute consistency of mathematics. No matter what method you use, you will always end up with the same result.

Let us begin.

Start by modelling the diagram on Q_1 of the Cartesian Coordinate Grid, Using B as the origin. For this method, we will only be calculating a quarter of the shaded area, then quadrupling at the end.
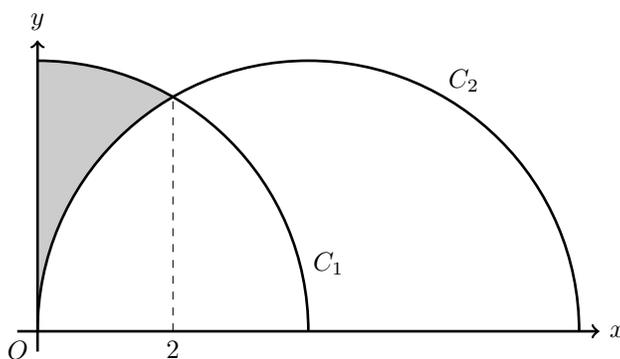
Let C_1 be $x^2 + y^2 = 16$, and C_2 be $(x-4)^2 + y^2 = 16$

Firstly, we prove that the curves meet at $x = 2$ by solving simultaneously (Even though you can tell just by looking).

$$x^2 + y^2 = (x-4)^2 + y^2$$
$$x^2 - (x-4)^2 = 0$$
$$x^2 - (x^2 - 8x + 16) = 0$$
$$8x = 16$$
$$x = 2$$

Now then, let us proceed by converting the curves into polar form using the following identities:

$$x^2 + y^2 = r^2, \quad x = r\cos\theta$$

Starting with C_1,

$$x^2 + y^2 = 16$$
$$r^2 = 16$$
$$C_1 : r = 4$$

And now C_2,

$$(x-4)^2 + y^2 = 16$$

$$x^2 + y^2 - 8x + 16 = 16$$

$$r^2 - 8r\cos\theta = 0$$

$$C_2 : r = 8\cos\theta$$

Now to covert the intersection point into polar coordinates (r, θ). Knowing that $x = 2$ and $r = 4$,

$$2 = 4\cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Therefore, the curves intersect at $(4, \frac{\pi}{3})$.

Finally, we will use the polar integration formula below to determine the area of the shaded region.

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$$

We will run the integral from $\theta = \frac{\pi}{3}$ to $\theta = \frac{\pi}{2}$ (to clip it off at the y axis).

$$\frac{1}{4}A = \frac{1}{2}\int_{\pi/3}^{\pi/2} \left(4^2 - (8\cos\theta)^2\right) \, d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{\pi/2} \left(16 - 64\cos^2\theta\right) \, d\theta$$

$$= 8\int_{\pi/3}^{\pi/2} \left(1 - 4\cos^2\theta\right) \, d\theta$$

Use the identity, $\quad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
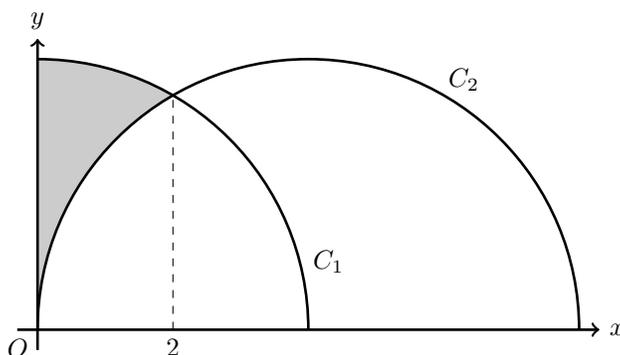
$$= 8\int_{\pi/3}^{\pi/2} \left(1 - 4 \times \frac{1 + \cos 2\theta}{2}\right) \, d\theta$$

$$= 8\int_{\pi/3}^{\pi/2} \left(-1 - 2\cos 2\theta\right) \, d\theta$$

$$= 8\left[-\theta - \sin 2\theta\right]_{\pi/3}^{\pi/2}$$

$$= 8\left[\left(-\frac{\pi}{2} - 0\right) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$$

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$$= 8 \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right)$$

$$= 4\sqrt{3} - \frac{4\pi}{3}$$

$$A = \boxed{16\sqrt{3} - \frac{16\pi}{3}}$$

Awesome.

3. Integration (Trig Sub)

For this method, we will reuse the previous model and graph, but integrate with respect to x.

$$C_1 : x^2 + y^2 = 16, \quad C_2 : (x-4)^2 + y^2 = 16$$

Lets begin by rearranging both equations to make y the subject.

$$C_1 : y = \sqrt{16 - x^2}, \quad C_2 : y = \sqrt{16 - (x-4)^2}$$

Now we integrate using the following formula for enclosed areas:

$$A = \int_{x_1}^{x_2} (\text{Upper Curve}) - (\text{Lower Curve}) \, dx$$

Where x_1 and x_2 are points of intersection. Proceeding now,

$$\frac{1}{4} A = \int_0^2 \sqrt{16 - x^2} - \sqrt{16 - (x-4)^2} \, dx$$

Separate the integrals and label them as follows.

$$I_1 = \int_0^2 \sqrt{16 - x^2} \, dx, \quad I_2 = \int_0^2 \sqrt{16 - (x-4)^2} \, dx$$
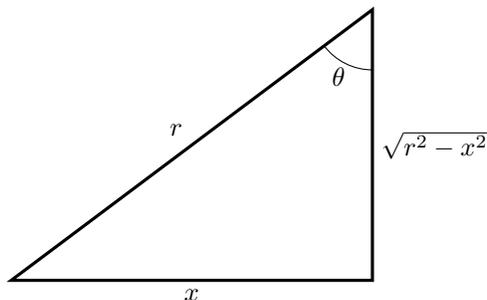
Notice that both of the integrals are in the form

$$I = \int \sqrt{r^2 - x^2} \, dx$$

Which has a standard well-known result. However, I will derive it for you anyways.

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For this integral we will need to use Trigonometric Substitution (Trig Sub).

Notice that $\sqrt{r^2 - x^2}$ reminds us of the Pythagorean Theorem, resembling one of the legs of a right angled triangle. Allow me to visualise this.

From this we can obtain that

$$x = r\sin\theta \quad \text{and} \quad r\cos\theta = \sqrt{r^2 - x^2}$$

To find our replacement for dx we simply differentiate the easier equation

$$dx = r\cos\theta\,d\theta$$

Now we substitute everything back into the integral.

$$\int \sqrt{r^2 - x^2}\,dx \quad \text{becomes} \int r\cos\theta \times r\cos\theta\,d\theta$$

And so,

$$I = r^2 \int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}r^2 \int 1 + \cos 2\theta\,d\theta$$

$$= \frac{1}{2}r^2(\theta + \frac{1}{2}\sin 2\theta) + C$$

Use the identity $\dfrac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

$$= \frac{1}{2}r^2(\theta + \sin\theta\cos\theta) + C$$

Now convert back to x using the equations we formed earlier.

$$I = \frac{1}{2}r^2 \left(\sin^{-1}\left(\frac{x}{r}\right) + \frac{x\sqrt{r^2 - x^2}}{r^2} \right) + C$$

$$= \boxed{\frac{1}{2}\left(r^2 \sin^{-1}\left(\frac{x}{r}\right) + x\sqrt{r^2 - x^2} \right) + C}$$

Now to apply the standard result to the integrals we had earlier

$$I_1 = \int_0^2 \sqrt{4^2 - x^2}\, dx$$

$$= \frac{1}{2}\left[16\sin^{-1}\frac{x}{4} + x\sqrt{16 - x^2}\right]_0^2$$

$$= \frac{1}{2}\left[16\sin^{-1}\left(\frac{1}{2}\right) + 2\sqrt{12}\right] - 0$$

$$= \frac{8\pi}{6} + 2\sqrt{3}$$

$$= \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$

$$I_2 = \int_0^2 \sqrt{16 - (x-4)^2}\, dx$$

$$= \left[\frac{x-4}{2}\sqrt{16 - (x-4)^2} + 8\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^2$$

$$= \left(\frac{-2}{2}\sqrt{16 - 4} + 8\sin^{-1}\left(-\frac{1}{2}\right)\right) - \left(\frac{-4}{2}\sqrt{16 - 16} + 8\sin^{-1}(-1)\right)$$

$$= \left(-\sqrt{12} - \frac{4\pi}{3}\right) - (0 - 4\pi)$$

$$= -2\sqrt{3} - \frac{4\pi}{3} + 4\pi$$

$$= \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$

Now, recall that

$$\frac{1}{4}A = I_1 - I_2$$

So finishing up,

$$\frac{1}{4}A = \frac{4\pi}{3} + 2\sqrt{3} - \left(\frac{8\pi}{3} - 2\sqrt{3}\right)$$

$$= 4\sqrt{3} - \frac{4\pi}{3}$$

$$A = \boxed{16\sqrt{3} - \frac{16\pi}{3}}$$

Conclusion

We have shown that the enclosed area can be expressed exactly as

$$A = 16\sqrt{3} - \frac{16\pi}{3}$$

No matter which method we use to obtain it. This article has essentially wrapped up how to disgustingly over-complicate a simple GCSE question.

Anyways, to the Year 11s reading this, try not to use integration to answer your GCSE question.

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Gian Zanel Sanchez Kadir (Y12)
DHSG
01/03/2026 23:37